

**1. The Lifetime of Stars (Lecture 10 material)**

Mass of $^1\text{H}$ nucleus	1.007825032 amu
Mass of $^4\text{He}$ nucleus	4.002603250 amu
Mass of $^{56}\text{Fe}$ nucleus	55.934941 amu
Avogadro's Number (amu/g)	$6.022 \times 10^{23}$
Speed of Light in vacuum, $c$	$2.9979 \times 10^8 \text{ m s}^{-1}$
Stefan-Boltzmann constant, $\sigma$	$5.67032 \times 10^{-8} \text{ J s}^{-1} \text{ m}^{-2} \text{ K}^{-4}$
Mean Radius of Sun, $R_{\odot}$	695950 km
Effective Surface Temperature of Sun, $T_e$	5800 K
Mass of Sun, $M_{\odot}$	$1.991 \times 10^{30} \text{ kg}$

- Calculate the energy production per hydrogen atom of the net proton burning reaction  $4 \text{ } ^1\text{H} \rightarrow \text{ } ^4\text{He}$ , express the answer in Joules [use SI units throughout so that  $E \text{ (J)} = m \text{ (kg)} c^2 \text{ (m s}^{-1})^2$ ]. What fraction of the total mass is converted to energy by this process?
- Calculate the energy production of the sun assuming steady-state and ignoring neutrino losses, so that production is equal to solar luminosity. Use the Stefan-Boltzmann law  $E \text{ (J s}^{-1} \text{ m}^{-2}) = \sigma T_e^4$  and assume the Sun is a sphere.
- Calculate the number of hydrogen atoms consumed by fusion in the sun per unit time. Then assume the Sun was initially made of pure hydrogen and calculate how long it could burn at its present luminosity before exhausting every hydrogen atom.
- Give at least three reasons why this is a gross overestimate of the life of the Sun.
- Try doing the same calculation for a supermassive main sequence star with surface temperature  $T_e = 35000 \text{ K}$  and luminosity  $L = 10^6 L_{\odot}$ . Assume the same average density as the sun so that  $M/M_{\odot} = (R/R_{\odot})^3$ . What is the longest such a star can burn hydrogen?
- What additional fraction of the initial mass of hydrogen could be converted to energy if the entire mass of the star were transformed from  $^4\text{He}$  into  $^{56}\text{Fe}$ ? How long could the star in part (e) maintain the same luminosity by this process?

**2. Synthesis of heavy elements by neutron capture (Lecture 10 material)**

- Draw a schematic chart of the nuclides for the elements shown in the following table (i.e.,  $46 \leq Z \leq 51$ ,  $56 \leq N \leq 74$ ). Leave the short-lived radioactive nuclei blank.
- Take it as given that  $^{108}\text{Pd}$  is on the s-process pathway. Trace the s-process pathway from  $^{108}\text{Pd}$  upwards. Identify all the nuclides that can be formed by the s-process.
- For elements Cd, In, Sn, and Sb only, identify and label all the nuclides that can be formed by the r-process. Assume the table lists all of the nuclides that might prevent the r-process from reaching these elements.
- Which isotopes of Cd, In, Sn, and Sb cannot be formed by either s-process or r-process? Do you observe anything systematic about their relative abundance?
- Separately* for Cd and for Sn, make a plot of the product of isotopic abundance (N) and neutron capture cross-section ( $\sigma$ ). The predictions of nuclear statistical equilibrium are: (i) that the s-process-only nuclides evolve to a constant product

$\sigma_N$ ; (ii) Nuclides with s-process and r-process contributions can be more abundant than this model; (iii) p-process-only nuclei can be lower than this value. Considering only s-process, r-process, and s+r-process nuclides, how consistent are the above data with these predictions for Cd? How about for Sn?

- f. Now, for Cd and for Sn *separately*, **assume** neutron capture equilibrium among the isotopes of a given element, so that one s-only isotope predicts the s-fraction for all isotopes of that element, and look at what is left over to estimate what fraction of the solar inventory of Cd and of Sn resulted from r-process nucleosynthesis.

Element	Stable or long-lived isotopes	Isotopic Abundance	Neutron Capture Cross section
<sup>46</sup> Pd	102		
	104		
	105		
	106		
	108		
	110		
<sup>47</sup> Ag	107		
	109		
<sup>48</sup> Cd	106	1.25%	0.653 barns
	108	0.89	0.478
	110	12.49	0.271
	111	12.80	0.631
	112	24.13	0.256
	113	12.22	0.708
	114	28.73	0.170
	116	7.49	0.106
<sup>49</sup> In	113	4.29%	
	115	95.71	
<sup>50</sup> Sn	112	0.97%	0.360 barns
	114	0.65	0.247
	115	0.34	0.0502
	116	14.54	0.0596
	117	7.68	0.306
	118	24.22	0.123
	119	8.59	0.200
	120	32.59	0.0365
	122	4.63	0.0273
	124	5.79	0.0349
<sup>51</sup> Sb	121	57.3%	
	123	42.7	

**3. Chemical thermodynamics is all about total and partial derivatives (this question goes back to Lecture 4 material)**

In the lecture notes you will find the following definitions:

$$dS = dq_{rev}/T$$

for reversible processes in one-component systems,  $dG = -SdT + VdP$ .

$$\alpha_P = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P, C_p \equiv \left( \frac{\partial q_{rev}}{\partial T} \right)_P = T \left( \frac{\partial S}{\partial T} \right)_P$$

a. Obtain an expression for the temperature change due to unit pressure change along a *reversible adiabat* (i.e., a path insulated from any heat flow,  $dq = dq_{rev} = 0$ ) for a one-component, one phase system.

Hints: you will need two facts from calculus, namely the equality of mixed partials

$$\left( \frac{\partial^2 A}{\partial B \partial C} \right) = \left( \frac{\partial^2 A}{\partial C \partial B} \right); A = A(B, C) \text{ and the cyclic rule } \left( \frac{\partial A}{\partial B} \right)_C \left( \frac{\partial B}{\partial C} \right)_A \left( \frac{\partial C}{\partial A} \right)_B = -1.$$

b. For the atmosphere, near sea level,  $T = 300 \text{ K}$  and  $C_p \sim 1000 \text{ J kg}^{-1} \text{ K}^{-1}$ . The mean molecular weight is  $\sim 29 \text{ g/mol}$ . You can get  $\alpha_P$  and density from the ideal gas law. Use your formula to estimate the temperature gradient (or “adiabatic lapse rate”) of the atmosphere. Does this seem about right for what happens to the temperature if you go up, say, a 1 km high mountain (note  $dP = \rho dz$ )?

c. On the other hand, near the top of mantle,  $T = 2000 \text{ K}$ ,  $\alpha_P = 2 \times 10^{-5} \text{ K}^{-1}$ , density =  $3300 \text{ kg m}^{-3}$ , and  $C_p = 1.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$ . Assuming the mantle is adiabatic, calculate the increase of temperature with unit pressure increase. The pressure at the base of the mantle is 138 GPa. If your temperature gradient obtains all the way down to core-mantle boundary, estimate the temperature there.

**4. Practice with phase diagrams, whoopee! (Lecture 4 material)**

Consider the MgO-SiO<sub>2</sub> binary phase diagram attached. Describe the sequence of events that occur (a qualitative description will suffice) during (a) equilibrium closed-system crystallization of a liquid with composition 50% MgO, 50% SiO<sub>2</sub> along the cooling path from 1900 to 1500 °C; (b) fractional crystallization of the same starting composition, where at any time the bulk composition of the system is the composition of the remaining liquid, and phases that have already crystallized cannot react with the liquid at lower temperature; (c) equilibrium closed-system melting of a rock initially consisting of 75% enstatite and 25% quartz (i.e. bulk composition = 70% SiO<sub>2</sub>) along the heating path from 1500 to 1900 °C; and (d) fractional melting of the same rock where at any time the bulk composition of the system is the composition of the residual solids, which cannot react with liquid already produced.

